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An independent set J_c of a graph G is called critical if

$$|J_c| - |N(J_c)| = \max\{|J| - |N(J)| : J \text{ is an independent set of } G\}$$

and a vertex subset U_c is called critical if

$$|U_c| - |N(U_c)| = \max\{|U| - |N(U)| = U \text{ is a subset of } V(G)\}.$$

In this paper, we will show that to find a critical independent set and a critical vertex subset of a graph are solvable in polynomial time.

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FINDING CRITICAL INDEPENDENT SETS AND CRITICAL VERTEX SUBSETS ARE POLYNOMIAL PROBLEMS

Cun-Quan Zhang*
Department of Mathematics
West Virginia University
Morgantown
West Virginia 26506 USA

ABSTRACT

An independent set J_c of a graph G is called critical if $|J_c|-|N(J_c)| = \max\{|J|-|N(J)|: J \text{ is an independent set of } G\}$, and a vertex subset U_c is called critical if $|U_c|-|N(U_c)| = \max\{|U|-|N(U)|: U \text{ is a vertex subset of } G\}$. In this paper, we will show that to find a critical independent set and a critical vertex subset of a graph are solvable in polynomial time.

Key words: Independent set, Polynomial algorithm
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§ 1. INTRODUCTION

It has been proved by mathematicians that finding a maximum independent set in some certain kinds of graphs is solvable in polynomial time (for example, line graphs, bipartite graphs, circle graphs, circular arc graphs and claw free graphs (see [GJ])). But it is well-known that it is an NP-complete problem for general graphs (see [GJS]). In this paper, we will investigate another problem -- finding a certain kind of independent sets in general graphs. An independent set J_c of a graph G is called critical if $|J_c|-|N(J_c)|$ is the maximum of $|J|-|N(J)|$ over all independent sets J of G, where N(J) is the set all vertices of G adjacent to some vertex of J. It will be proved in this paper that finding a critical independent set of a graph is solvable in polynomial time. Let

$$\alpha_c = \max\{|J|-|N(J)| : J \text{ is an independent set of } G\}$$

which is a parameter of a graph G and is called the critical independence number of G. The critical independence number α_c of a graph plays the central role in the study of fractional independence functions and fractional matching functions of graphs [GZ]. (It is proved in [GZ] that the fractional independence number and the fractional matching number of a graph G are $\frac{n-\alpha_c}{2}$ and $\frac{n+\alpha_c}{2}$, respectively, where $n=|V(G)|$).

Some related problems and parameters of graphs will also be investigated in this paper. A vertex subset U_c of a graph G=(V,E) is called critical if $|U_c|-|N(U_c)|$ is the maximum of $|U|-|N(U)|$ over all vertex subsets U of G. Let

$$\mu_c = \max\{|U|-|N(U)| : U \text{ is a vertex subset of } G\}$$

which is a parameter of a graph G. Some similar parameters of graphs have been studied by Woodall [WD] and Mohar [MB].

The binding number b(G) [WD] and the isoperimetric number i(G) [MB] of a graph G are defined as the following:

$$b(G) = \min \left\{ \frac{|N(U)|}{|U|} : U \subset V(G)\ ,\ \ U \neq \emptyset \text{ and } N(U) \neq V(G) \right\} ;$$

$$i(G) = \min \left\{ \frac{|\partial(U)|}{|U|} : U \subset V(G)\ ,\ \ U \neq \emptyset \text{ and } |U| \leq \frac{|V(G)|}{2} \right\} .$$

(where $\partial(U)$ is the number of edges of G with one endvertex in U). Later in this paper, it will be proved that $\alpha_c = \mu_c$.

Since the empty set is an independent set and the set of all vertices of a graph G is also a vertex subset of G, it is trivial that

$$\alpha_c \geq 0 \text{ and } \mu_c \geq 0$$

for any graph G. Note that the empty set and the entire graph are critical vertex subsets of some connected graph G if $\mu_c(G)=0$. If we are to avoid these two trivial vertex subsets $\emptyset$ and V(G), we may consider the following parameter of a graph G:

$$\mu_c' = \max \{ |U| - |N(U)| \ :\ U \subset V(G),\ U \neq \emptyset \text{ and } U \neq V(G) \}.$$

But for any connected graph G and a vertex v of G, we have that $N(V(G)\setminus\{v\}) = V(G)$ and therefore the parameter $\mu_c'(G)$ still have a lower bound -1. In order to get more information about graphs, we prefer to only consider those vertex subsets U of a graph G such that $U \neq \emptyset$ and $N(U) \neq V(G)$ which is similar to the definition of the binding number of graphs. A vertex subset U of G is called <u>proper</u> if $U \neq \emptyset$ and $N(U) \neq V(G)$. A proper vertex subset U_{pc} of G is called <u>critical</u> if $|U_{pc}| - |N(U_{pc})|$ is the maximum of $|U| - |N(U)|$ over all proper vertex subsets U of G. The parameter μ_{pc} of a graph G is defined as the following:

$$\mu_{pc} = \max \{ |U| - |N(U)| \ :\ U \subset V(G),\ U \neq \emptyset \text{ and } U \neq V(G) \}$$

The problems will be proved to be solvable in polynomial time in this paper are listed as the following:

INSTANCE. let G=(V,E) be a graph with the vertex set V and the edge set E and k be an integer.

PROBLEM 1. Is there an independent set J of G such that

$$|J|-|N(J)| \geq k?$$

PROBLEM 1*. Find a critical independent set J_c and the critical independence number α_c of G. (That is to find

$$\alpha_c = |J_c|-|N(J_c)|$$
$$= \max\{|J|-|N(J)| : J \text{ is an independent set of } G\}).$$

PROBLEM 2. Is there a vertex set U of G such that

$$|U|-|N(U)| \geq k?$$

PROBLEM 2*. Find a critical vertex subset U_c and the parameter μ_c of G. (That is to find

$$\mu_c = |U_c|-|N(U_c)|$$
$$= \max\{|U|-|N(U)| : U \text{ is a vertex subset of } G\}).$$

PROBLEM 3. Is there a proper vertex subset U of G such that

$$|U|-|N(U)| \geq k?$$

PROBLEM 3*. Find a critical proper vertex subset U_{pc} and the parameter μ_{pc} of G (That is to find

$$\mu_{pc} = |U_{pc}|-|N(U_{pc})|$$
$$= \max\{|U|-|N(U)| : U \subset V(G),\ U \neq \emptyset \text{ and } U \neq V(G)\}).$$

§ 2. MAIN RESULTS

THEOREM 1.

Problem 3 and 3* are solvable in polynomial t.me.

Before we prove the Theorem 1 we would like to consider the following problems first. And the Theorem 1 will be a corollary of the Theorem 2.

INSTANCE. Let $G=(V,E)$ be a graph with the vertex set V and the edge set E, $\{u,v\}$ be an ordered pair of nonadjacent vertices of G and k be an integer.

PROBLEM 4. Is there a vertex subset U of G such that

$$u \in U, \quad v \notin N(U) \text{ and}$$

$$|U|-|N(U)| \geq k?$$

PROBLEM 4*. Find a vertex subset U_o of G such that

$$|U_o|-|N(U_o)|$$

$$= \max \{ |U|-|N(U)| : U \subset V(G),\ u \in U \text{ and } v \notin N(U) \}.$$

The vertex subset U_o found in problem 4* is called (u,v)-critical subset of G.

THEOREM 2.

The problems 4 and 4* are solvable in polynomial time.

The following lemmas will be used in the proof of the Theorem 2.

LEMMA 3. (Hall's Theorem [HP])

Let $B=(V_1,V_2;\ E)$ be a bipartite graph. The graph B has a matching covering all vertices of V_2 if and only if $|U| \leq |N(U)|$ for any subset U of V_2.

LEMMA 4.

Let $B = (V_1,\ V_2;\ E)$ be a bipartite graph. Assume that there is no matching of B covering all vertices of V_2. We will have the following conclusions:

(i). There is a subset U of V_2 such that $|N(U)| < |U|$;

(ii). Let U_0 be a subset of V_2 such that $|U_0| - |N(U_0)|$ is as great as possible, then there is a matching of the induced bipartite subgraph $(U_0, (N(U_0);\ E[U_0,N(U_0)])$ covering all vertices of $N(U_0)$.

Note, if A and B are a pair of disjoint vertex subset of a graph G, the set of all edges joining A and B is denoted by $E(A,B)$.

PROOF

The conclusion of (i) is an immediate corollary of Hall's Theorem.

Let U be a subset of V_2 such that $|U| - |N(U)|$ is as great as possible. Let $B'=(U,N(U); E[U,N(U)])$ be the subgraph of B induced by $U \cup N(U)$. We claim that $|X| \leq |N(X) \cap U|$ for any subset X of $N(U)$. If not, let $X \subseteq N(U)$ such that

$$|X| > |N(X) \cap U|.$$

We will consider the subset $Y = U\backslash N(X)$. Note that

$$N(Y) = N(U\backslash N(X)) \subseteq N(U)\backslash X.$$

And

$$\begin{aligned} |Y| - |N(Y)| &\geq |U\backslash N(X)| - |N(U)\backslash X| \\ &= [|U| - |U \cap N(X)|] - [|N(U)| - |X|] \\ &= [|U| - |N(U)|] + [|X| - |U \cap N(X)|] \\ &> |U| - |N(U)|. \end{aligned}$$

This contradicts the choice of U that $|U| - |N(U)|$ is maximum. So by Hall's theorem, there is a matching in B' which covers all vertices of $N(U)$.

###

PROOF OF THE THEOREM 2.

We are only to prove that problem 4* is solvable in polynomial time. Let $G=(V,E)$ be a graph with the vertex set $V=\{1,2,3,...,n\}$ and the edge set E. We will consider the ordered pair of vertices (1,2) of G and find a (1,2)-critical vertex subset.

Define a bipartite graph $B = (X,Y; E_B)$ where

$$X = \{x_1, \ldots, x_n\},$$

$$Y = \{y_1, \ldots, y_n\}$$

and $\quad E_B = \{(x_i, y_j):\ (i,j) \text{ is an edge of } G\}.$

Let V' be a subset of V(G), then the corresponding subsets in X and Y are denoted by X(V') and Y(V'), respectively. For example, if $V'=\{i_1, \ldots, i_t\}$, then $X(V')=X(\{i_1, \ldots, i_t\})=\{x_{i_1}, \ldots, x_{i_t}\}$ and $Y(V')=Y(\{i_1, \ldots, i_t\})=\{y_{i_1}, \ldots, y_{i_t}\}$. (Here X and Y can be considered as bijections mapping $\{1,2,\ldots,n\}$ onto $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$). If $W=\{x_{i_1}, \ldots, x_{i_t}\} \subset X$ (or $W=\{y_{i_1}, \ldots, y_{i_t}\} \subset Y$), then the corresponding subset $\{i_1, \ldots, i_t\}$ of V(G) is denoted by $X^{-1}(W)$ (or $Y^{-1}(W)$, respectively). The set of all neighbors of a vertex u in B is denoted by $N_B(u)$. If i is a vertex of G, then

$$N_B(x_i) = \{y_j \in Y:\ (x_i, y_j) \in E_B\} = \{y_j \in Y:\ (i,j) \in E(G)\} = Y(N(i)).$$

A weight $w: X \cup Y \to [0,2]$ is called a (1,2)-<u>proper weight</u> of B if

$$w(x_1)=2,$$
$$w(y_2)=1,$$
$$1 \le w(x_i) \le 2 \qquad \text{for each vertex } x_i \in X,$$
$$0 \le w(y_i) \le 1 \qquad \text{for each vertex } y_i \in Y$$

and $$0 \le w(x_i)+w(y_i) \le 2 \qquad \text{for each edge } (x_i, y_i) \in E_B.$$

The <u>total weight</u> $\sum_{u \in X \cup Y} w(u)$ of B is denoted by w(B). A (1,2)-proper weight w_m of B is call <u>optimum</u> if

$$w_m(B) = \max\{w(B):\ w \text{ is a (1,2)-proper weight of } B\}.$$

It is obvious that finding an optimum (1,2)-proper weight of B is a linear programing problem. Hence it is solvable in polynomial time. The purpose of the investigation of an optimum (1,2)-proper weight w_m of B is to prove that the vertex subset $\{i \in V(G):\ w_m(x_i) > 1\}$ is a (1,2)-critical subset of G.

I. Let w_m be an optimum (1,2)-proper weight of B and U_o be a (1,2)-critical vertex subset of G. We claim that

$$w_m(B) \geq 2n + |U_o| - |N(U_o)| = 2n + \alpha_c \tag{1}$$

Consider the following weight w_1 of B:

$$w_1(x_i) = \begin{cases} 2 & \text{if } i \in U_o \\ 1 & \text{otherwise} \end{cases}$$

$$\text{and } w_1(y_j) = \begin{cases} 0 & \text{if } j \in N(U_o) \\ 1 & \text{otherwise} \end{cases} .$$

It is easy to see that w_1 is a (1,2)-proper weight of B and

$$w_1(B) = 2|U_o| + (n-|U_o|) + (n-|N(U_o)|)$$
$$=2n + |U_o| - |N(U_o)|.$$

By the choice of w_m, we have verified the inequality (1).

II. Let
$X_b = \{x_i: w_m(x_i) > 1\}$
$Y_s = \{y_i: w_m(y_i) < 1\}$
$X_{b'} = \{x_i: 1 < w_m(x_i) < 2\}$
and $Y_{s'} = \{y_i: 0 < w_m(y_i) < 1\}$.

By the definition of (1,2)-proper weight , it is obvious that $N_B(X_b) \subseteq Y_s$ and $N_B(Y_{s'}) \subseteq [X \backslash X_b] \cup X_{b'}$.

III. Case 1. Suppose that there is a matching M in the induced subgraph $B(X_{b'} \cup Y_{s'})$ covering all vertices of $Y_{s'}$. We claim that X_b is a (1,2)-critical vertex subset in this case. We are to adjust the weight w_m so that the new weight of each vertex in $X_{b'}$ is two and the new weight of each vertex in $Y_{s'}$ is one. If we can verify that this new weight is optimum, by the inequality (1), it can be shown that $X^{-1}(X_b)$ is a (1,2)-critical vertex set of G.

If (u,v) is an edge of M, then let u=M(v) and v=M(u). And the sets of vertices of $X_{b'}$ covered and not covered by M is denoted by $M(Y_{s'})$ and $X_{b'} \backslash M$, respectively. Thus $X_{b'} \cup Y_{s'} = Y_{s'} \cup M(Y_{s'}) \cup (X_{b'} \backslash M)$ since M covers all vertices of $Y_{s'}$.

Consider the following weight w_2 of B:

$$w_2(v) = \begin{cases} 2 & \text{if } v \in X_{b'} \\ 0 & \text{if } v \in Y_{s'} \\ w_m(v) & \text{otherwise} \end{cases} .$$

Since any vertex of Y adjacent to a vertex x_i of $X_{b'}$ must be in Y_s in which the weight of each vertex is zero, w_2 is a (1,2)-proper weight of B. Note that w_m is optimum and the total weight $w_m(B)$ cannot be less than $w_2(B)$. We claim that w_2 is also an optimum (1,2)-proper weight of B by proving that $w_m(B) \le w_2(B)$. Since

$$\{v \in X \cup Y: \ w_m(v) \neq w_2(v)\} = X_{b'} \cup Y_{s'},$$

we must have that

$$\begin{aligned} & w_m(B) - w_2(B) \\ &= \sum_{v \in X_{b'}} [w_m(v) - w_2(v)] + \sum_{v \in Y_{s'}} [w_m(v) - w_2(v)] \\ &= \sum_{v \in Y_{s'}} \{[w_m(v) - w_2(v)] + [w_m(M(v)) - w_2(M(v))]\} + \\ &\qquad + \sum_{v \in X_{b'} \setminus M} [w_m(v) - w_2(v)]. \end{aligned}$$

Here

$$\begin{aligned} & [w_m(v) - w_2(v)] + [w_m(M(v)) - w_2(M(v))] \\ &= [w_m(v) - w_m(M(v))] - [w_2(v) - w_2(M(v))] \\ &\le 2 - (0+2) \\ &= 0 \end{aligned}$$

for any $v \in Y_{s'}$, and

$$w_m(v) - w_2(v) < 2-2 = 0$$

for any $v \in X_{b'} \setminus M$. This implies that $w_m(B) - w_2(B) \le 0$. Therefore w_2 is also an optimum (1,2)-proper weight of B and $w_m(B) = w_2(B)$. The total weight of w_2 is

$$w_2(B) = 2|X_b| + |X \setminus X_b| + |Y \setminus Y_s|.$$

Since $N_B(X_b) \subseteq Y_s$, we must have that

$$\begin{aligned} w_m(B) = w_2(B) \le \ & 2|X_b| + (n - |X_b|) + (n - |N(X_b)|) \\ &= 2n + |X_b| - |N(X_b)| \\ &= 2n + |X^{-1}(X_b)| - |N(X^{-1}(X_b))| \\ &\le 2n + \alpha_c \quad \text{(by the definition} \end{aligned}$$

of α_c).

By (1), all equalities hold and therefore $X^{-1}(X_b) = \{i \in V(G): w_m(x_i) > 1\}$ is a (1,2)-critical vertex subset of G.

IV. Case 2. If there is no matching of $B(X_{b'} \cup Y_{s'})$ covering all vertices of $Y_{s'}$. By Lemma 4, there is a subset Y_o of $Y_{s'}$ such that (i). $|Y_o| > |N_B(Y_o) \cap X_{b'}|$ and, (ii). there is a matching M' in the induced bipartite subgraph $B(Y_o \cup [N_B(Y_o) \cap X_{b'}])$ covering all vertices of $N_B(Y_o) \cap X_{b'}$. We are to adjust the weight w_m so that the new weight of each vertex in $Y_o \cup [N_B(Y_o) \cap X_{b'}]$ is one. We will find that the new weight is greater than w_m. It will contradicts that w_m is optimum.

Consider the following weight w_3:

$$w_3(v) = \begin{cases} 1 & \text{if } v \in Y_o \cup [N_B(Y_o) \cap X_{b'}] \\ w_m(v) & \text{otherwise} \end{cases}.$$

The weight w_3 is (1,2)-proper since $x_1 \notin N(Y_{s'})$ and any vertex adjacent to a vertex of Y_o must be in $[X \backslash X_b] \cup [N_B(Y_o) \cap X_{b'}]$ in which the weight of each vertex is one. Note that w_m is an optimum (1,2)-proper weight of B, thus the total weight of w_m cannot be less than the total weight of w_3. Since

$$\{v \in V(B): w_m(v) \neq w_3(v)\} \subseteq Y_o \cup [N_B(Y_o) \cap X_{b'}]$$
$$= [N_B(Y_o) \cap X_{b'}] \cup M'[N_B(Y_o) \cap X_{b'}] \cup [Y_o \backslash M'],$$

where $M'[N_B(Y_o) \cap X_{b'}]$ and $[Y_o \backslash M']$ are the sets of vertices of Y_o covered and not covered by M', respectively. Thus we must have that

$$w_m(B) - w_3(B)$$
$$= \sum_{v \in N_B(Y_o) \cap X_{b'}} [w_m(v) - w_3(v)] + \sum_{v \in Y_o} [w_m(v) - w_3(v)]$$
$$= \sum_{v \in N_B(Y_o) \cap X_{b'}} \{[w_m(v) - w_3(v)] + [w_m(M(v)) - w_3(M(v))]\} +$$
$$+ \sum_{v \in Y_o \backslash M'} [w_m(v) - w_3(v)].$$

But

$$[w_m(v) - w_3(v)] + [w_m(M(v)) - w_3(M(v))]$$

$$= [w_m(v)-w_m(M(v))] - [w_3(v)-w_3(M(v))]$$
$$\leq 2 - (1+1)$$
$$= 0$$

for any $v \in N_B(Y_o) \cap X_{b'}$, and

$$w_m(v)-w_3(v) < 1-1 = 0$$

for any $v \in Y_o \backslash M'$. Since $|Y_o| > |N_B(Y_o) \cap X_{b'}|$, the set $Y_o \backslash M'$ is not empty and we have that $w_m(B)-w_3(B) < 0$. This contradicts that w_m is optimum and completes the proof of the theorem.

###

Proof of the Theorem 1.

For any ordered pair of non-adjacent vertices $\{u,v\}$ of G, by Theorem 2, we can find a (u,v)-critical subset in polynomial time. The (u,v)-critical subset is denoted by $U_{C(u,v)}$. Then choose a (u_o,v_o)-critical vertex subset $U_{C(u_o,v_o)}$ such that

$$|U_{C(u_o,v_o)}| - |N(U_{C(u_o,v_o)})|$$
$$= \max\{|U_{C(u,v)}| - |N(U_{C(u,v)})| : (u,v) \text{ are an ordered pair of nonadjacent vertices of } G\}$$

which is a proper critical subset of G desired in problem 3*. The total cost of finding a critical proper vertex subset is polynomial since the cost of finding a (u,v)-critical vertex subset is polynomial and the number of pairs of non-adjacent vertices in G is at most $\binom{n}{2}$.

###

Theorem 5.

Problem 2 and 2* are solvable in polynomial time.

Proof.

Let G=(V,E) be a graph. Consider a new graph G' by adding two isolated vertices x, y to G. Let U_c be a (x,y)-

critical subset of G'. Obviously, $x, y \in U_c$ and it is clear that $U_c \setminus \{x,y\}$ is a critical subset of G.

Alternating proof of the Theorem: Define a bipartite graph $B = (X,Y;E_B)$ where $X=\{x_1,...,x_n\}$, $Y=\{y_1,...,y_n\}$ and $E_B=\{(x_i,y_j): (i,j)$ is an edge of $G\}$. And assign a weight w to the vertex set of B such that $w: X \cup Y \to [0,2]$ and

$$1 \le w(x_i) \le 2 \qquad \text{for each vertex } x_i \in X,$$
$$0 \le w(y_i) \le 1 \qquad \text{for each vertex } y_i \in Y$$
and $$0 \le w(x_i)+w(y_i) \le 2 \qquad \text{for each edge } (x_i,y_i) \in E_B.$$

The total weight $\sum_{u \in X \cup Y} w(u)$ is denoted by $w(B)$. Let

$$w_m(B) = \max\{w(B): w \text{ is a weight of B satisfies the above definition}\}.$$

By an argument similar to the proof of Theorem 2, we can prove that the set $V_b=\{i \in V(G): w_m(x_i)>1\}$ is an critical vertex subset of G.

###

Theorem 6.

Problem 1 and 1* are solvable in polynomial time.

Before the proof of the Theorem 6, we will prove a Theorem by which Theorem 6 is an immediate corollary.

Theorem 7.

Let $G=(V,E)$ be a graph.

(i). Let U_c be a critical vertex subset of G and $T_1,...,T_h$ be all non-trivial components of the induced subgraph $G(U_c)$. Then $J=V(U_c)\setminus[V(T_1) \cup ... \cup V(T_h)]$ is a critical independent set of G and $|J|-|N(J)| = |U_c|-|N(U_c)|$.

(ii). max $\{|J|-|N(J)| : J$ is an independent set of $G\}$

$= \max\{|U|-|N(U)| : U$ is a vertex subset of $G\}$.

That is, any critical independent set is also a critical vertex subset of G and therefore $\alpha_c = \mu_c$.

Proof.

It is obvious that

$$0 \le \alpha_c = \max\{|J|-|N(J)| : J \text{ is an independent set of } G\}$$
$$\le \max\{|U|-|N(U)| : U \text{ is a vertex subset of } G\} = \mu_c \qquad \ldots\ldots.(2)$$

since any independent set is a vertex subset of G.

Let U_c be a critical vertex subset of G. The Theorem is trivial if U_c is an empty set. Assume that U_c is a counterexample to the Theorem containing minimum number of vertices. By the assumption U_c cannot be an independent set of G. Let T be a non-trivial component of the induced subgraph $G(U_c)$. It is clear that $V(T) \subseteq N(T)$ since T is not a singleton. Thus

$$|U_c \setminus T| - |N(U_c \setminus T)| \ge [|U_c|-|T|] - [|N(U_c|-|T|]$$
$$= |U_c| - |N(U_c)|$$

It implies that $U_c \setminus T$ is also a critical vertex subset of G which contains less number of vertices than U_c. It contradicts the choice of U_c and therefore completes the proof of the Theorem.

###

Proof of the Theorem 6.

Let U_c be a critical vertex subset of G. Let $T_1,\ldots,T_t$ be all non-trivial components of the induced subgraph $G(U_c)$. Then by Theorem 7, $U_c \setminus \{T_1,\ldots,T_t\}$ is a critical independent

set of G. Since finding a critical vertex subset U_c and deleting all vertices of U_c incident with some edge of $G(U_c)$ only need polynomial cost, finding a critical independent set is solvable in polynomial time.

Alternating proof of the Theorem (also see [GZ]):

Consider a weight $w: V(G) \to [0,1]$ such that

$0 \le w(v) \le 1$ for each vertex v of G

and $w(u)+w(v) \le 1$ for each edge (u,v) of G

The total weight $\sum_{u \in V(G)} w(u)$ is denoted by w(G). Let

$$w_m(G) = \max\{w(G): w \text{ is a weight of B satisfies the above definition}\}.$$

Obviously finding w_m is a linear programing problem. By an argument similar to the proof of Theorem 2, we can prove that the set $V_b = \left\{ v \in V(G) : w_m(v) > \frac{1}{2} \right\}$ is an critical independent set of G.

(Note that the weight w defined above is called a fractional independence function of a graph G which was introduced by Domke, Hedetniemi, Laskar in [DHL] and by Grinstead, Slater in [GS2], and were studied by Grinstead, Slater in [GS1] and by Zhang in [CG].)

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Note, S. Poljak (personal communication) recently suggested an alternative proof of Theorem 2 by the theory of integer programing.

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